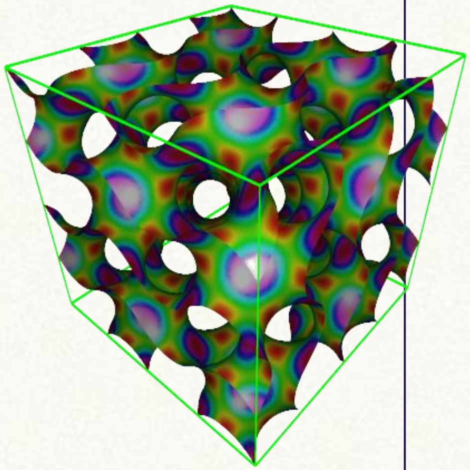
Minimal Surfaces Make for Cost Effective Statues



Yanni Bills

WHAT ARE MINIMAL SURFACES?

HEURISTICALLY DEFINED TERMS

Euclidian Space

A two- or three-dimensional space in which Euclid's postulates apply

1. A straight line segment can be drawn joining any two points
2. Any straight line segment can be extended indefinitely in a straight line
3. Given any straight line segment, a circle can be drawn having the segment be its radius, and one endpoint its center
4. All right angles are congruent
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

Postulates courtesy of Math 113 – Harvard University – Fall 1997, Prof. Curtis T. McMullen

Curvature

Curvature κ denotes how much a curve deviates from being a straight line

For example

The curve on the left has less curvature than that on the right

The notion of curvature for surfaces can also be thought of as deviation from a flat plane. However, the realization of curvatures for surfaces is a bit more complex.

Mean Curvature

An *extrinsic* measure that describes the local curvature of a surface that sits within some ambient space. It can be thought of as the average of the curvatures along principal direction planes

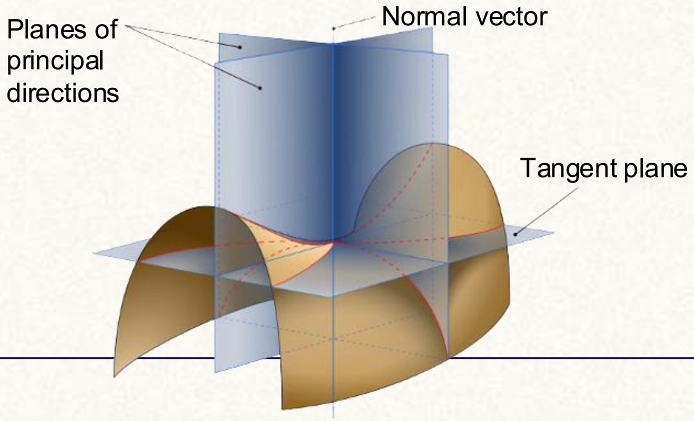


Diagram courtesy of Wikipedia "Principal curvature"

Weingarten Map

In its matrix form, the Weingarten Map describes a linear transformation $L: T_P M -> T_P M$. It can be thought of as the derivative of the Gauss Map – a mapping from any point on a surface to the unit sphere S^2.

For example, if $n: M \to S^2$ is our Gauss Map (for surface M), then our derivative $Dn(x)$ at any point $p \in M$ is a linear map

$$Dn(p): T_p M -> T_{n(p)}(S^2)$$

where the codomain is again $T_p M$.

SHOWCASE BY EXAMPLE

The Helicoid

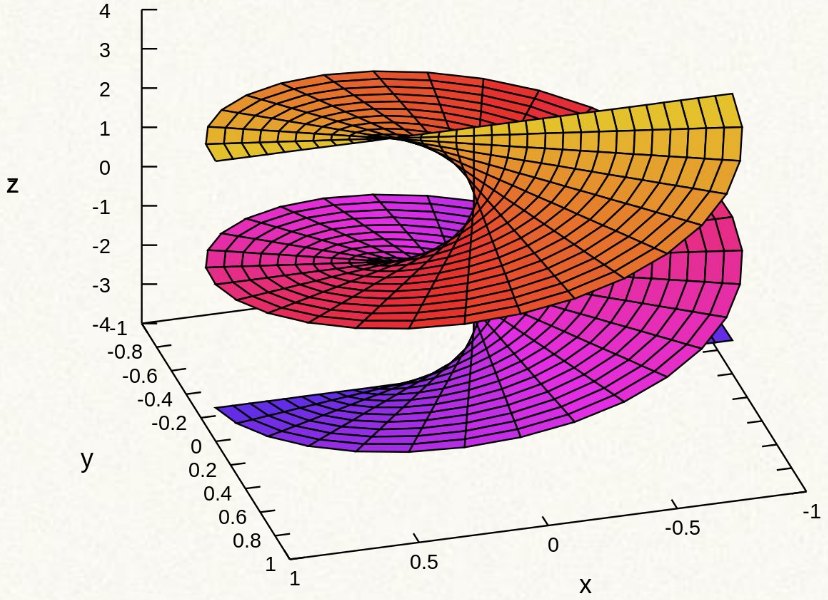


Diagram courtesy of Wolfram MathWorld

The helicoid can be parameterized like so

$$x(u^1, u^2) = (u^2 \cos u^1, u^2 \sin u^1, u^1)$$

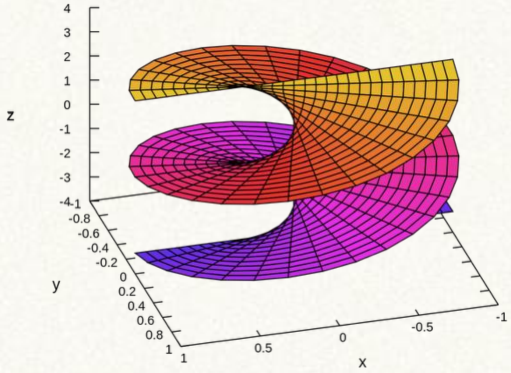
And then

$$x_1 = (-u^2 \sin u^1, u^2 \cos u^1, 1)$$
$$x_2 = (\cos u^1, \sin u^1, 0)$$

Allows for

$$g_{ij} = \langle x_i, x_j \rangle$$

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$



Our First Fundamental Form

With similar computations, we continue by

$$x_{11} = (-u^2 \cos u^1 , -u^2 \sin u^1 , 0)$$
$$x_{12} = x_{21} = (-\sin u^1 , \cos u^1 , 0)$$
$$x_{22} = (0,0,0)$$

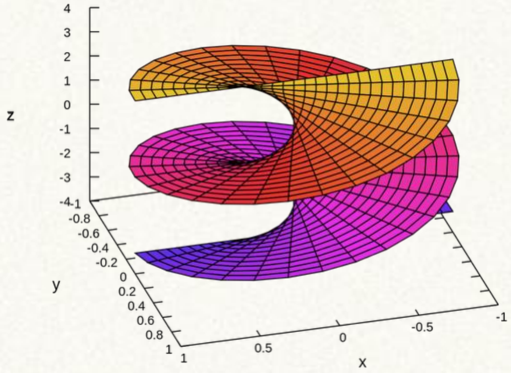
We then generate

$$\vec{n} = \frac{x_1 \times x_2}{\|x_1 \times x_2\|}$$

Which helps us find

$$L_{ij} = \langle x_{ij}, \vec{n} \rangle$$

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$



Our Second Fundamental Form

With a relatively tedious set-up, we find

$$\frac{1}{\det(g_{ij})}\begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} L_1^1 & L_2^1 \\ L_1^2 & L_2^2 \end{pmatrix}$$

Which is our Weingarten Map L

The Eigenvalues of this matrix are our principal curvatures

k_1 and k_2

Computing our mean curvature H can be done as follows

$$H = \frac{1}{2}(k_1 + k_2)$$

OR

$$H = \frac{1}{2}\mathrm{tr}(L)$$

In the case of our Helicoid

$$H = 0$$

Meaning our helicoid, and other surfaces with this trait, would be defined as *minimal*

Geometric Meaning?

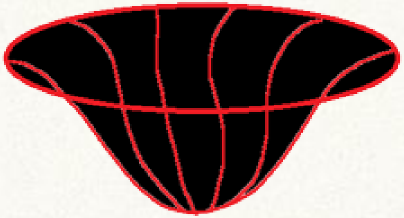
Locally Area Minimizing Surfaces

Drawing a sufficiently small loop on a minimal surface creates a small surface bounded by that loop

That small surface is the area minimizing surface for that loop

Perturbing that surface even a little bit would only increase the area of that little surface

Area-Minimizing



Not

Area-Minimizing

Greater Implications

It is worthwhile to also define the notion of Gaussian curvature K

Using what we've already discussed, we have that

$K = k_1 k_2$ or $K = \det(L)$

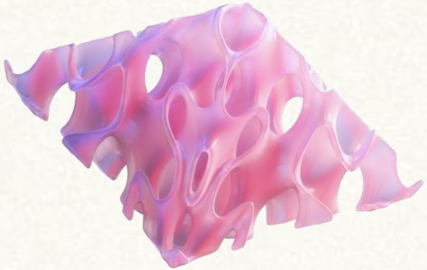
But for a minimal surface, we have that $H = 0$

So, $0 = \frac{1}{2}(k_1 + k_2)$

Implying that, $0 = k_1 + k_2$ and either

$0 = k_1 = k_2$ *or* k_1 and k_2 have opposite signs

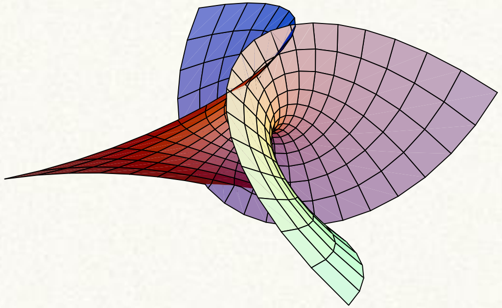
Meaning that $K \leq 0$



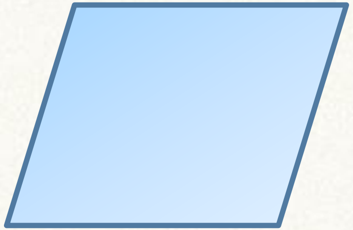
Figures Z1 and Z2. A blue and pink gyroid. Adapted from "Beauty of Minimal Surfaces" by D. Patie, 2022, Behance. Retrieved from https://www.behance.net/gallery/139866467/Beauty-of-minimal-surfaces-01.

WHAT DO THEY LOOK LIKE?

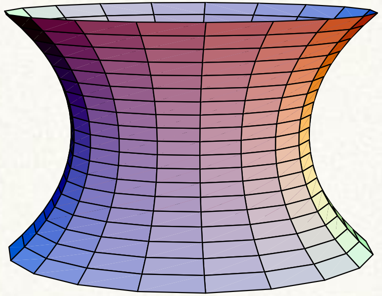
CONCRETE EXAMPLES



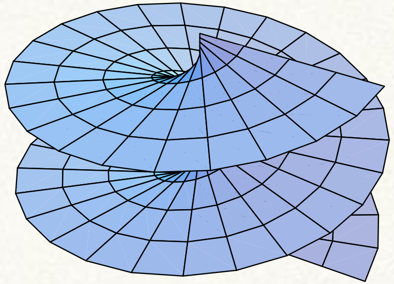
Enneper Surface



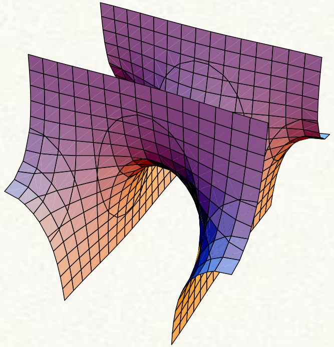
Plane



Catenoid



Helicoid



Scherk's doubly periodic surface

Diagrams courtesy of Wolfram MathWorld

MORE COMPLICATED EXAMPLES

LAWSON-KLEIN BOTTLE

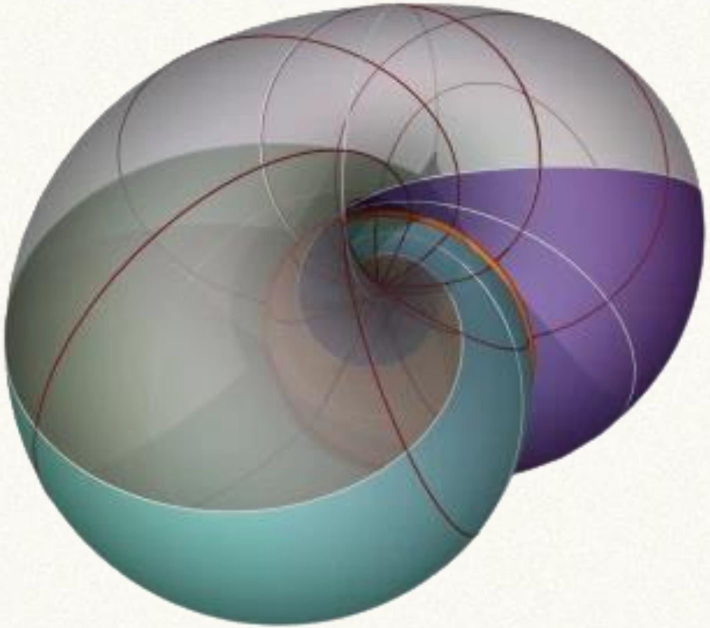
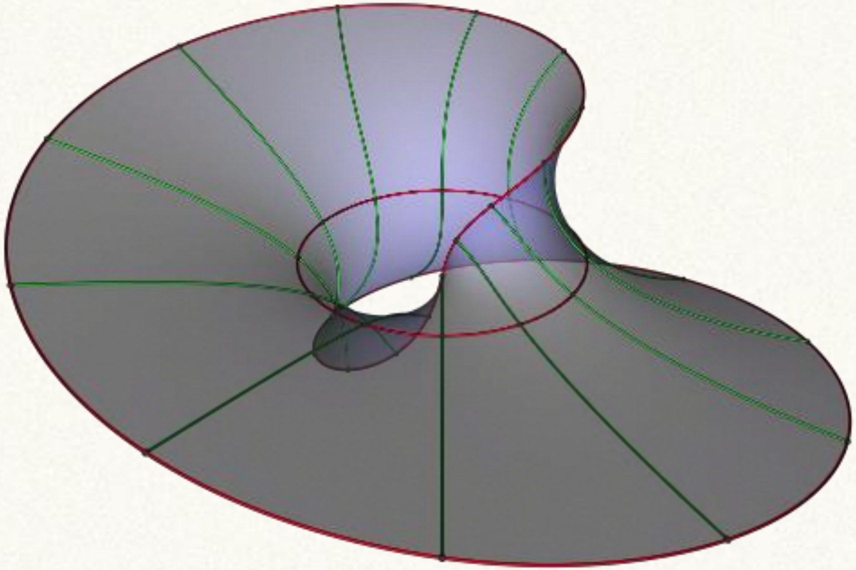


Figure LK. A Lawson-Klein Bottle. Adapted from "Lawson's Klein Bottle (Annuli IV)" by M. Weber, 2016, The Inner Frame. Retrieved from https://theinnerframe.org/2016/02/08/lawsons-klein-bottle-annuli-iv/

This surface is resultant of stereographic projection from the hypersphere of Lawson's surface whose topology is a Klein bottle. A Klein bottle is a non-orientable surface with no boundary and no notion of inside vs outside.

MEEKS' MÖBIUS STRIP



A self intersecting, minimal surface in Euclidian space possessing a circular boundary. It possesses the topology of an open Möbius strip. This was first described by *William Hamilton Meeks III*

Courtesy of https://minimal.sitehost.iu.edu/archive/Bjoerling/Bjoerling/moebius/web/index.html

MÖBIUS STRIPS AND BJÖRLING

Each higher dimensional minimal surface with the topology of a Möbius strip can be described and created via something called the *Björling Problem*.

This problem effectively describes a unique minimal surface from its boundary curve and tangent planes along this curve.

It can be solved as such

Let $c(s)$ be a real analytic curve in $\mathbb{R}^3$ defined over an interval *I*, with $c'(s) \neq 0$ and a vector field $n(s)$ along c such that $\|n(t)\| = 1$ and $c'(t) \cdot n'(t) = 0$. Then, the following surface is minimal:

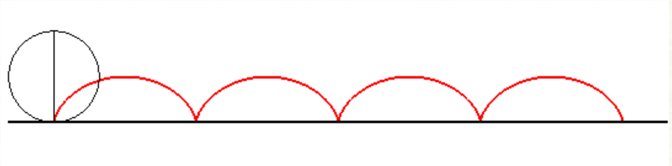
$$X(u,v) = \mathrm{Re}\left(c(w) - \mathrm{i} \int_{w_0}^{w} n(w) \times c'(w)\, \mathrm{d}w \right)$$

with $w = u + \mathrm{i}v \in \Omega, u_0 \in I, and\ I \subset \Omega$ is a simply connected domain where the interval is included and the power series expansion of $c(s)$ and $n(s)$ are convergent.

RESULTS OF BJÖRLING'S PROBLEM

Certain curves result in certain minimal surfaces when put through Björling's Problem

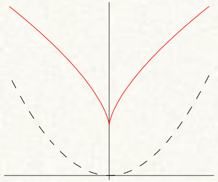
Cycloid Curve



Yields

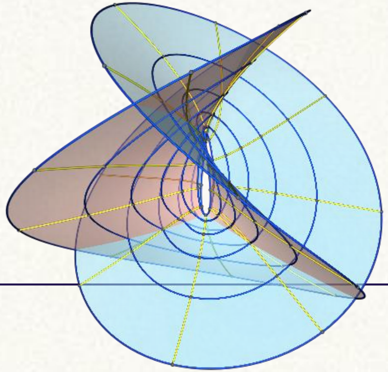
Catalan's Minimal Surface

Semicubical Parabola



Yields

The Henneburg Surface



GYROIDS

The gyroid is infinitely connected (containing no straight lines), and one of the most aesthetically interesting minimal surfaces to behold. The gyroid was discovered by NASA's own Alan Schoen over 50 years ago. Its aestheticism comes from its nature to divide space into two mirrored labyrinths of passages.

Following this, one will understand why it is that the word "*beauty*" is associated with minimal surfaces.

The images below were rendered, and even animated, by Remi Causse and Damien Patie. Of course, note that some creative liberties were taken for the sake of making breathtaking artwork.

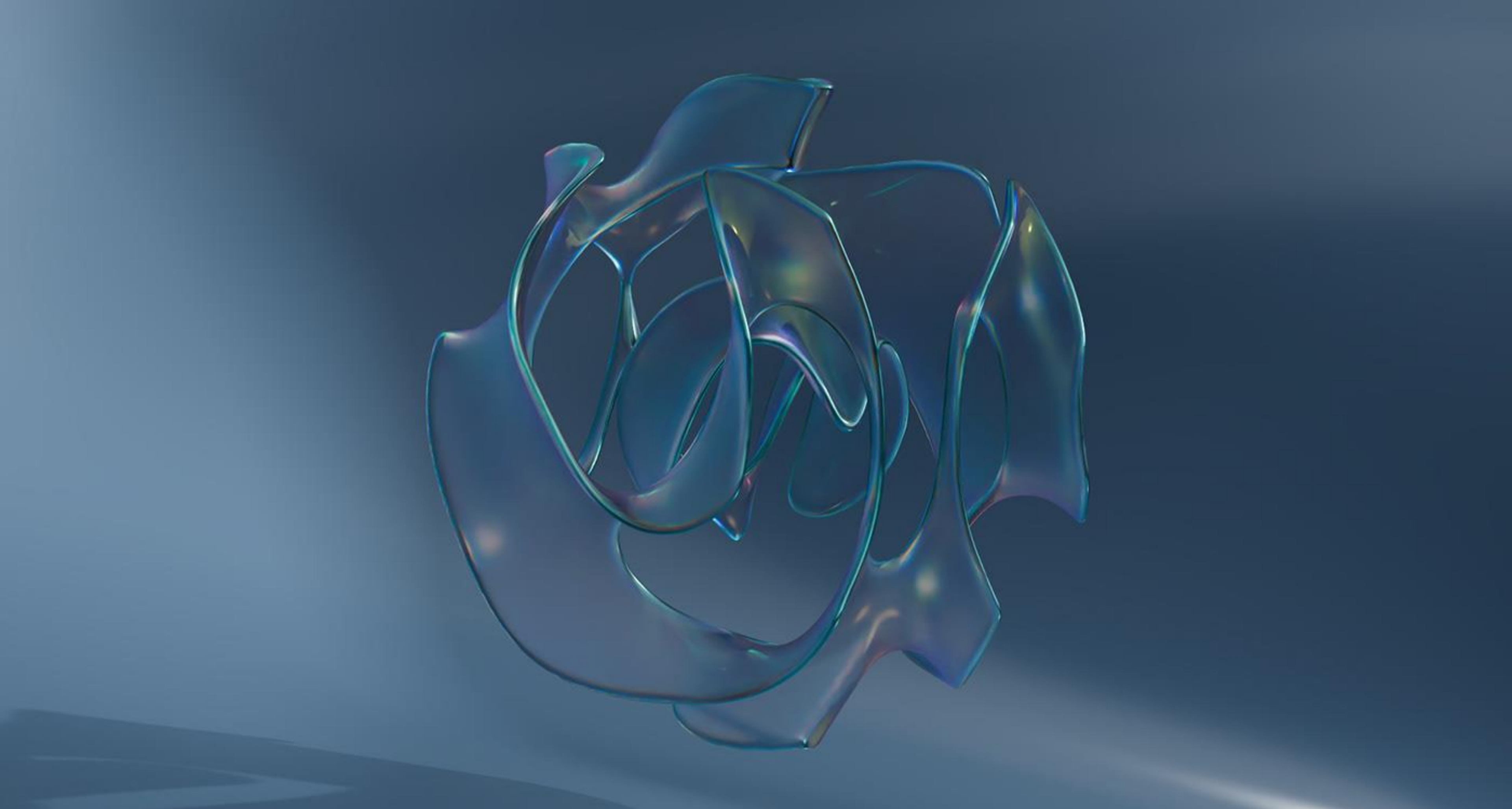


Figure 1. A transparent gyroid. Adapted from "Beauty of Minimal Surfaces" by D. Patie, 2022, Behance. Retrieved from https://www.behance.net/gallery/139866467/Beauty-of-minimal-surfaces-01.

Figure 2. A green gyroid. Adapted from “Beauty of Minimal Surfaces” by D. Patie, 2022, Behance. Retrieved from https://www.behance.net/gallery/139866467/Beauty-of-minimal-surfaces-01.

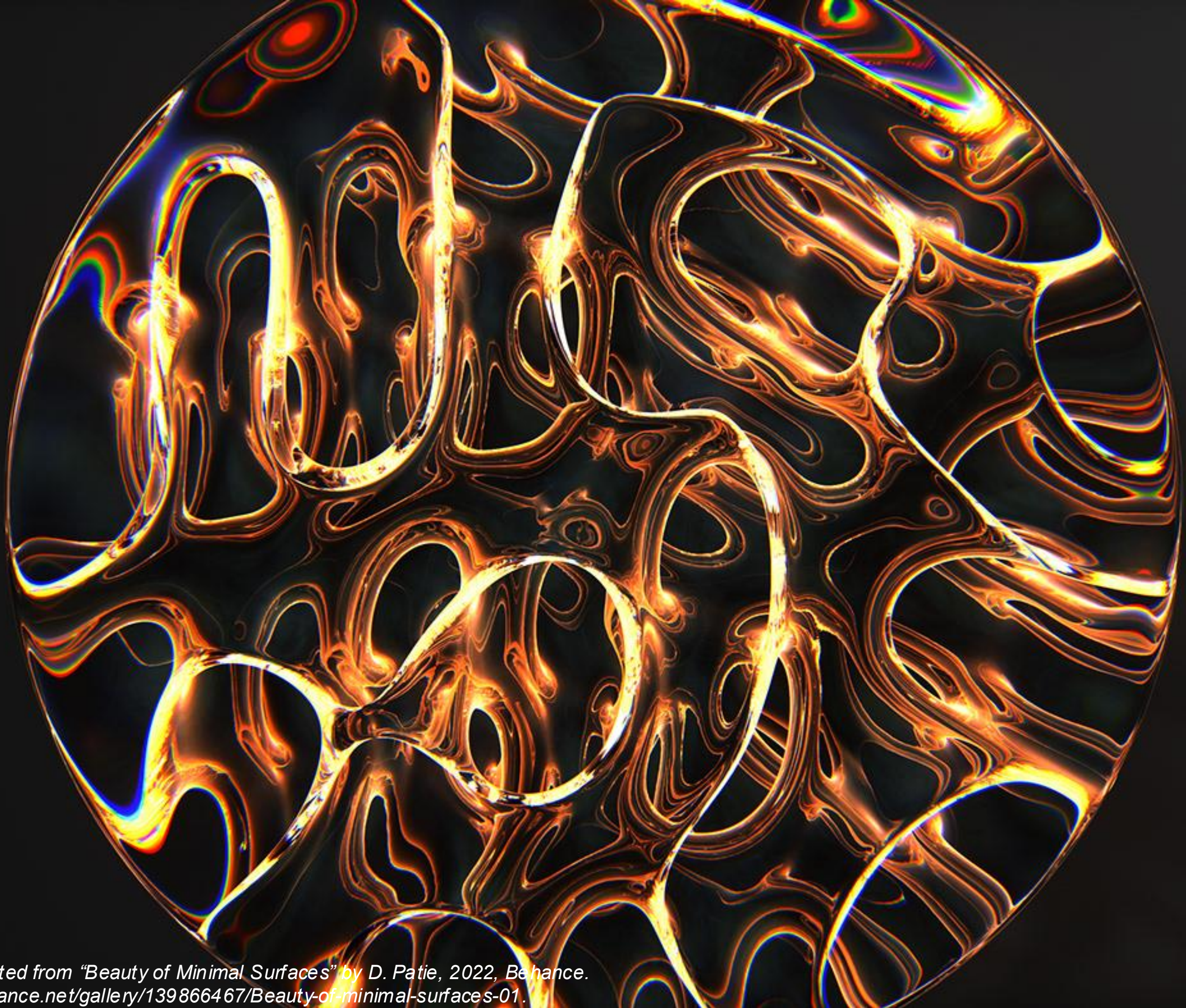


Figure 3. A golden gyroid. Adapted from “Beauty of Minimal Surfaces” by D. Patie, 2022, Behance. Retrieved from https://www.behance.net/gallery/139866467/Beauty-of-minimal-surfaces-01.

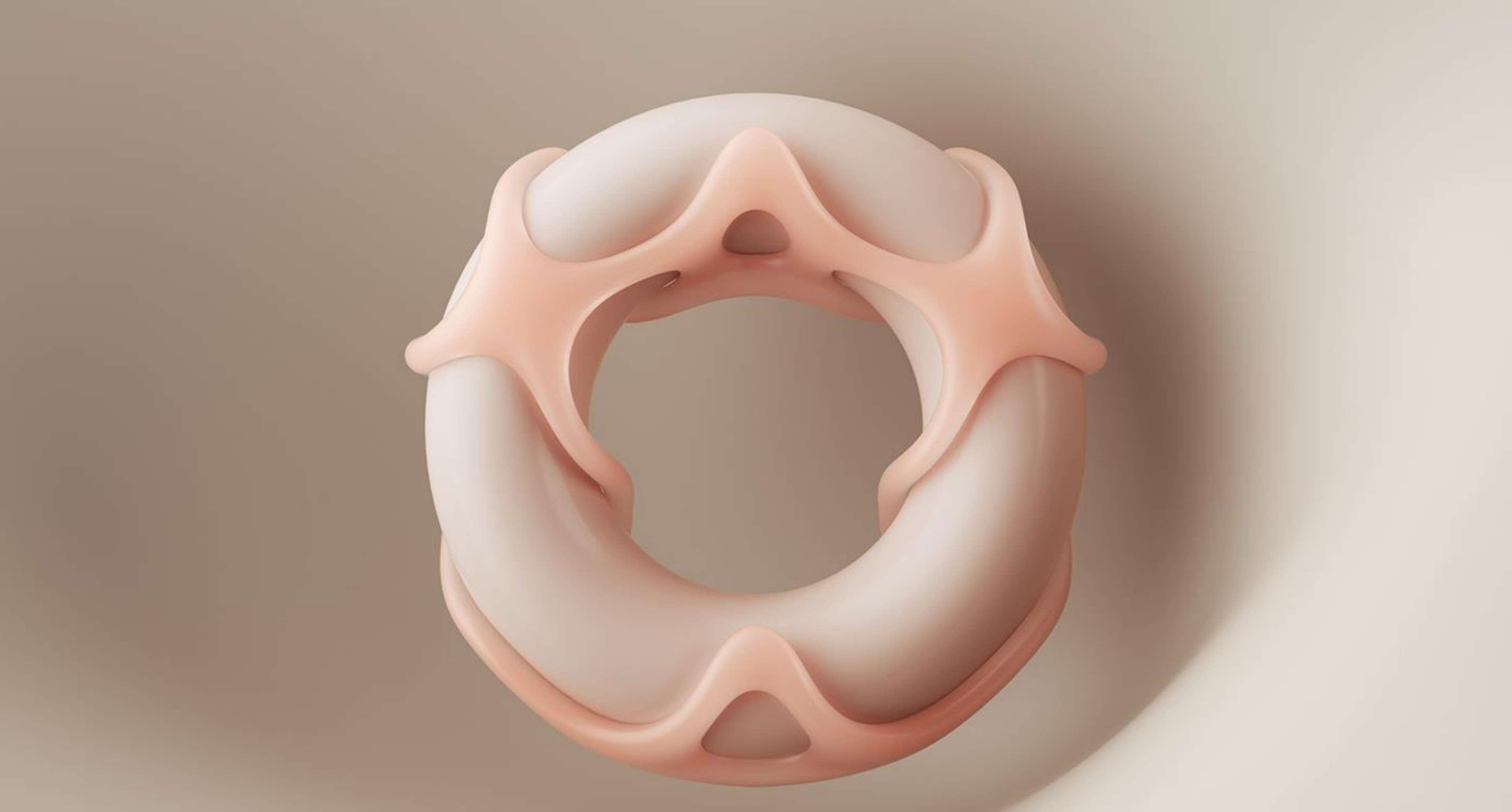


Figure 4. A torus-like gyroid. Adapted from "Beauty of Minimal Surfaces" by D. Patie, 2022, Behance. Retrieved from https://www.behance.net/gallery/139866467/Beauty-of-minimal-surfaces-01.

Figure 5. A dark gyroid. Adapted from "Beauty of Minimal Surfaces" by D. Patie, 2022, Behance. Retrieved from https://www.behance.net/gallery/139866467/Beauty-of-minimal-surfaces-01.

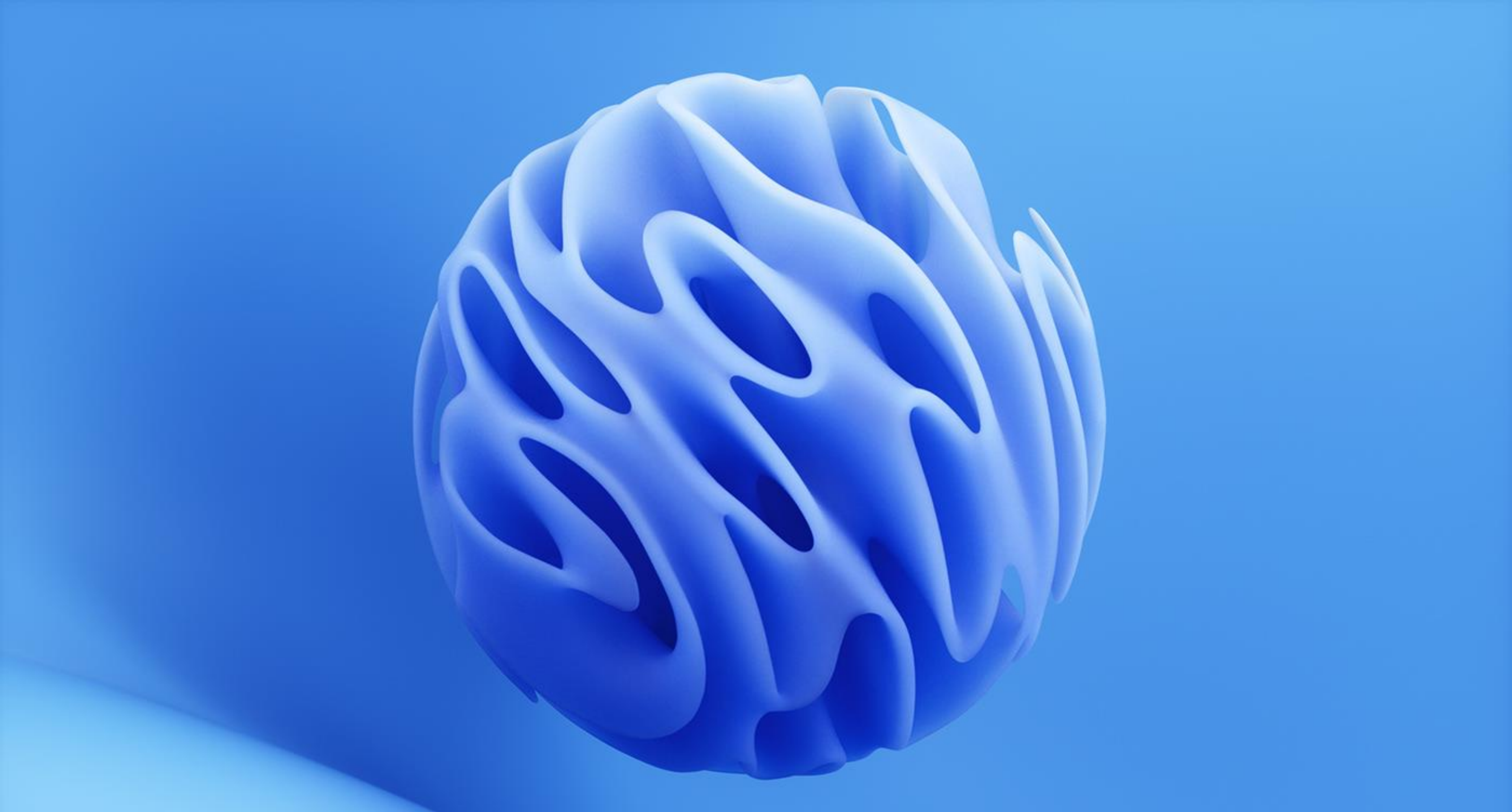


Figure 5. A blue gyroid. Adapted from “Beauty of Minimal Surfaces” by D. Patie, 2022, Behance. Retrieved from https://www.behance.net/gallery/139866467/Beauty-of-minimal-surfaces-01.

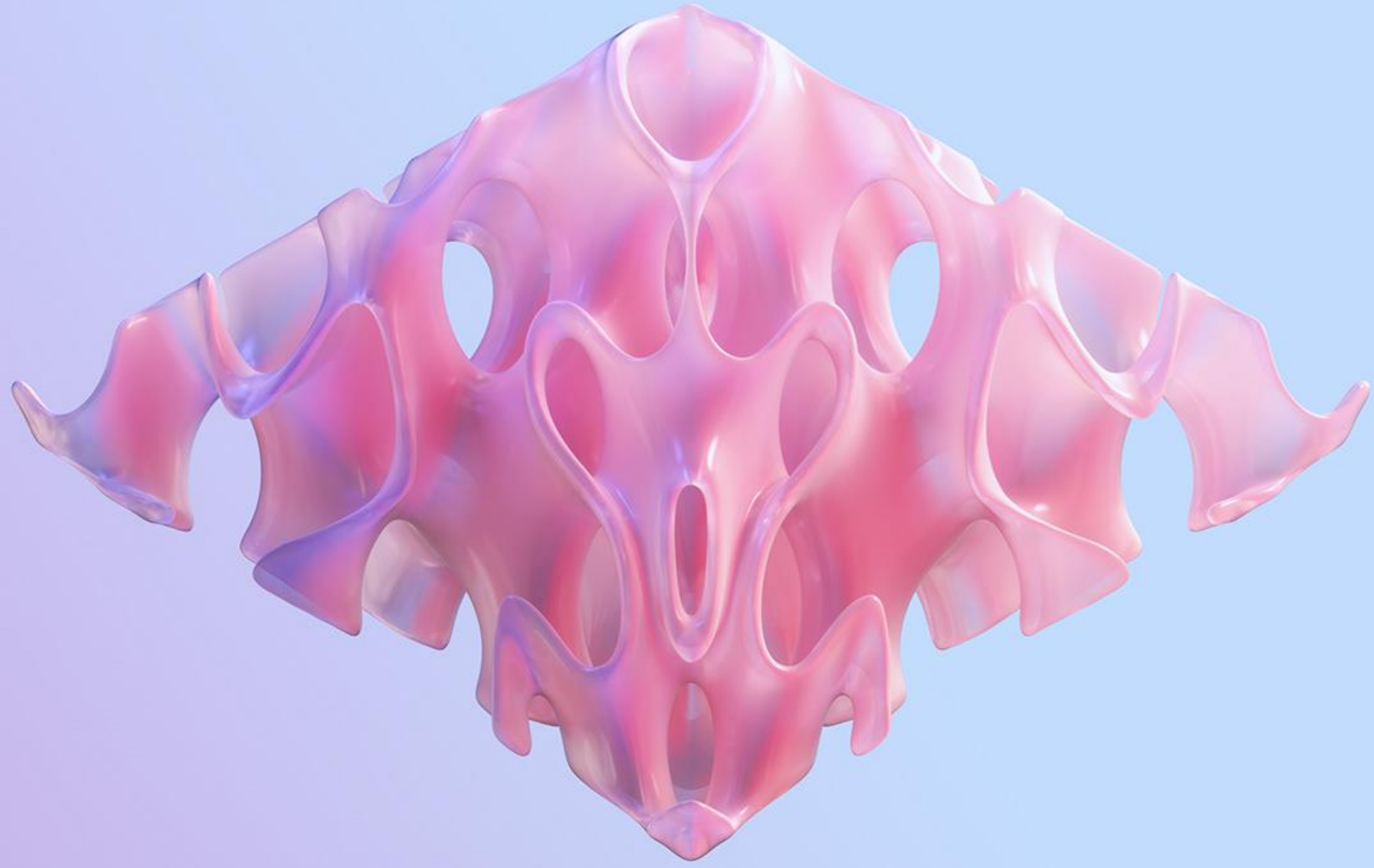


Figure 7. A pink gyroid. Adapted from “Beauty of Minimal Surfaces” by D. Patie, 2022, Behance. Retrieved from https://www.behance.net/gallery/139866467/Beauty-of-minimal-surfaces-01.

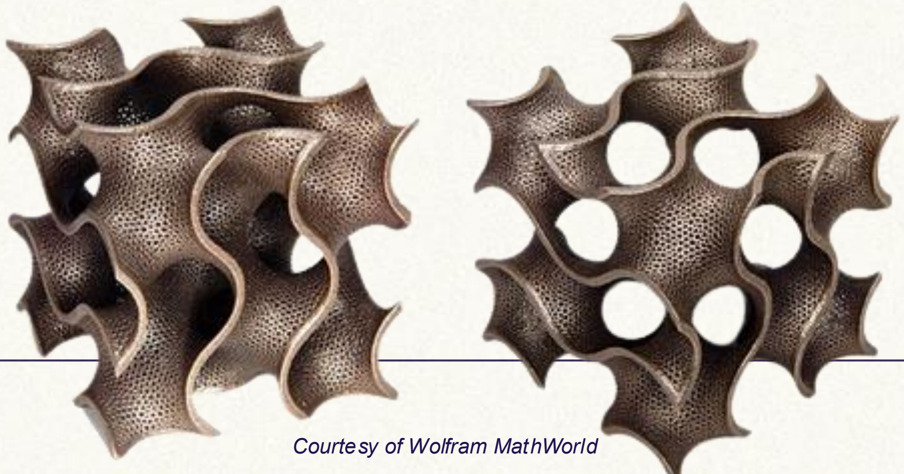
Figure 8. A pink gyroid's right arm. Adapted from "Beauty of Minimal Surfaces" by D. Patie, 2022, Behance. Retrieved from https://www.behance.net/gallery/139866467/Beauty-of-minimal-surfaces-01.

THE GYROID'S WONDROUS QUIRKS

The gyroid is triply periodic, meaning that a small piece of the surface can be used to construct the entire surface through the translation of copies in three, independent directions.

This is made evident by its simplistic trigonometric approximation

$$\cos x \sin y + \cos y \sin z + \cos z \sin x = 0$$



Courtesy of Wolfram MathWorld

ADDRESSING THE TITLE

AREA MINIMIZATION

As mentioned before, minimal surfaces are *locally* area-minimizing.

However, the entirety of the minimal surface itself doesn't have to be area minimizing.

So what does it mean when I've insinuated that minimal surfaces make for *cost-effective* statues?

The Title is a Lie

There is no consistent reason to believe that statues use less material, and cost less, when modeled to look like minimal surfaces.

But this lie was not perpetuated for no reason. In fact, referring to it now creates the perfect segue to a final topic of interest.

THE STORY OF PLATEAU'S PROBLEM

Plateau's problem effectively acts as the framework of finding the surface with minimal area enclosed by a given boundary curve in 3-space

In 1849, a blind physicist by the name of Joshua Plateau showcased that a minimal surface could be created by dipping a wire frame into a soapy solution.

In 1967, an architect by the name of Frei Otto famously utilized this framework to design a lightweight, yet expansive covering for a structure in Montreal

Despite this problem's name, the notion of finding the minimal surface for a given curve had initially been posed by Euler and Lagrange in 1760.
Since surface tension is proportional to area and energy is such to surface tension, there was a case to be made about finding *energy*-minimizing surfaces.

PLATEAU'S PROBLEM: MATHEMATICALLY

One of the more discussed generalizations of the problem is formulated as such

Given a simply closed curve Γ in $\mathbb{R}^3$, determine a mapping $x\colon \overline{D} \to \mathbb{R}^3$ where

$\mathrm{D} = \{(u, v) \in \mathbb{R}^2 | u^2 + v^2 < 1\}$ *is the open unit disk and*

$x(u, v) = (x^1(u, v), x^2(u, v), x^3(u, v))$ *such that*

(i) $x \in C^2(D) \cap C^0(D)$;

(ii) $\Delta x = x_{uu} + x_{vv} = 0, \mathrm{E} = x_u^2 = x_v^2 = G$, *and* $F = x_u \cdot x_v = 0$ on D;

(iii) x *maps* ∂D *homeomorphically onto* Γ.

But this doesn't quite capture the essence of Plateau's problem.

THE PEAK OF PLATEAU'S PROBLEM

Plateau's problem deals with some questions of importance, with some of these being existence, uniqueness, and regularity of solutions.

And in its most pure essence,

It is the desire to find a minimal surface within a given boundary Γ.

Throughout the 20th century, numerous mathematicians would find solutions to Plateau's problem for certain variants of Γ.

And to this day, mathematicians strive to solve for more generalizations of this problem.

CONNECTIONS TO THE PHYSICAL

As mentioned in the Plateau Problem's introduction, Plateau found a way to physically describe minimal surfaces using soap films.

The existence of minimal surfaces in the physical world goes to show that nature does have a tendency to be frugal.

Nature, if you'd like to personify it as a being with a taste for art styles, is minimalist. Nature prefers surfaces with locally area minimizing behavior to that of the non-minimalistic way of humankind.

But for the sake of brevity – and to avoid getting too philosophical – let us explore this idea by example.

MINIMAL SURFACES IN NATURE

Soap Film From Bubbles

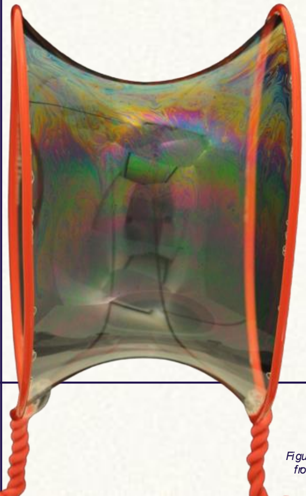
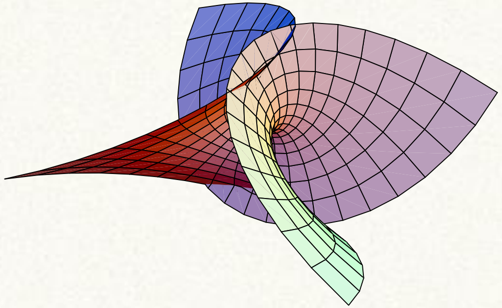


Figure HS. A helicoid bubble. Adapted from "Soap Films on a Spiral" by T. Kinsman, Science Photo Library, Retrieved from https://www.sciencephoto.com/media/705228/view/soap-films-on-a-spiral, Copyright: Ted Kinsman/SCIENCE PHOTO LIBRARY

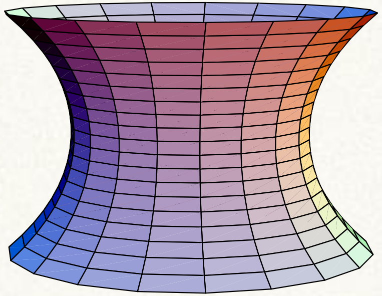
Evergreen Shrub Leaves

Figure EL. Evergreen shrub. Adapted from "Snowy South Manchester" by Manchester Evening News, 2013, Retrieved from https://www.manchestereveningnews.co.uk/incoming/gallery/snowy-south-manchester-790974

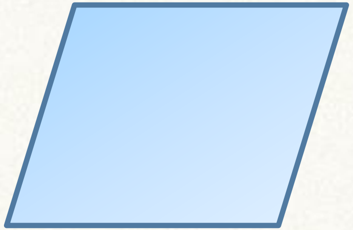
REMEMBER THESE?



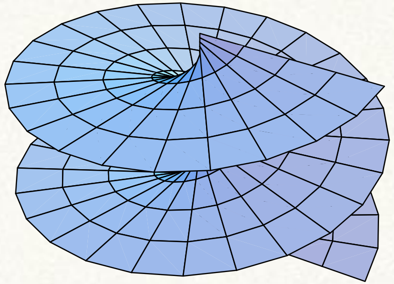
Enneper Surface



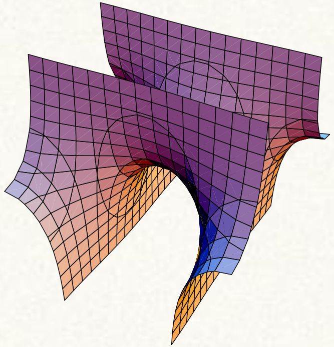
Catenoid



Plane



Helicoid



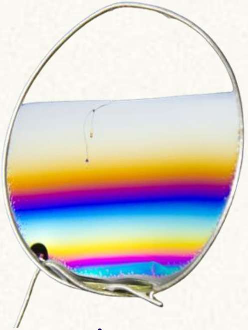
Scherk's doubly periodic surface

Diagrams courtesy of Wolfram MathWorld

LOOK AGAIN

Figure ES. An Enneper bubble. Adapted from "Soap films" by H. Segerman, 2015, Retrieved from https://www.youtube.com/watch?v=jReQUm9EB9k

Enneper Surface

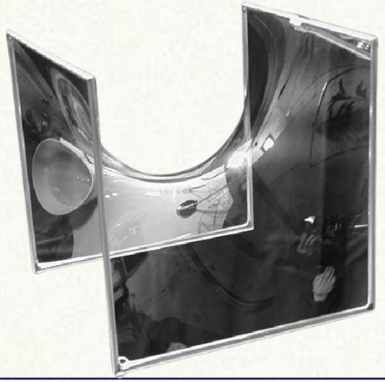


Plane

Catenoid

Helicoid

Figure HS. A helicoid bubble. Adapted from "Soap Films on a Spiral" by T. Kinsman, Science Photo Library, Retrieved from https://www.sciencephoto.com/media/705228/view/soap-films-on-a-spiral, Copyright: Ted Kinsman/SCIENCE PHOTO LIBRARY



Scherk's doubly periodic surface

Figure SS. A Scherk's bubble. Adapted from "Forms with no Name" by A. Capanna, 2011, ResearchGate, Retrieved from https://www.researchgate.net/figure/Basento-river-soap-blow-bubble-check-it-is-the-minimal-surface-set-from-this-peculiar_fig2_264238453

ARCHITECTURE AND STATUES

AT THIS POINT, IT'S A FORMALITY

Sequin, Carlo. *Cube-Volution-5.* 2009, Bronze, Schneider Museum of Art, Ashland

Foster, Michael. *Trefoil Waves.* Figured Ash, Breezy Hill Turning, Denver

Thiemicke, Oskar. *Catalan's minimal surface (pavilion concept).* 2021, Bronze, Behance, Dessau

I don't expect that any of these are quite cost-effective. Rather, I'd expect the contrary, given their need for complex forging or woodwork techniques.

This marks the extent of my knowledge on minimal surfaces

THANK YOU

Ucla × UCLA UMC UCLA Undergraduate Math Council

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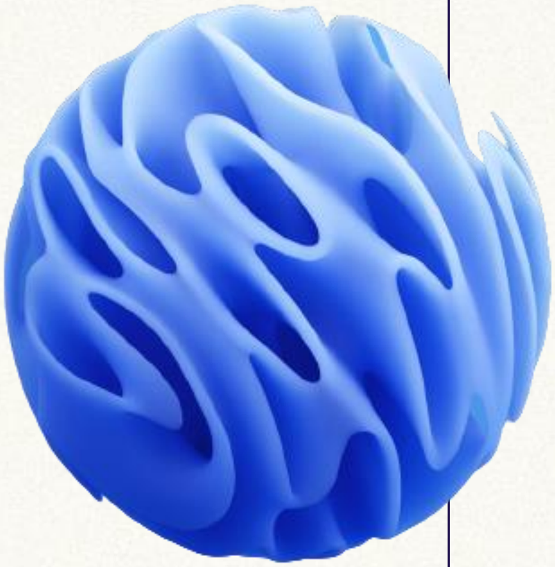

Figure 5. A blue gyroid. Adapted from "Beauty of Minimal Surfaces" by D. Patie, 2022, Behance. Retrieved from https://www.behance.net/gallery/139866467/Beauty-of-minimal-surfaces-01.

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